

MATHEMATICAL ANALYSIS – SUMS

AIMS

Grasp an intuition of some of the objects studied in mathematical analysis, such as *sums* and *series*. For this, we start here with finite sums, to be approached in a creative way. A separate module deals with the concept of series.

RESOURCES

PowerPoint, paper, pen, coloured pencils, and some vivid intuition.

Each activity needs to be properly introduced by a detailed discussion, and perhaps guided to favour the intuition of some of the students. Each activity takes at least 15 mins, and time for additional discussions has to be taken into account.

In any case, the time needed for each activity may considerably vary according to the interests of the class and the specific skills of the students.

The material and the activities presented in this tutorial are extremely challenging, so it is important to follow the students, to stimulate and encourage them, and no one has to be discouraged if (s)he does not grasp the ideas at once.

CONTEXT

Summing consecutive numbers.

Algebraic and geometric approaches.

Cancellations.

Sums

Summing numbers is sometimes very tricky.

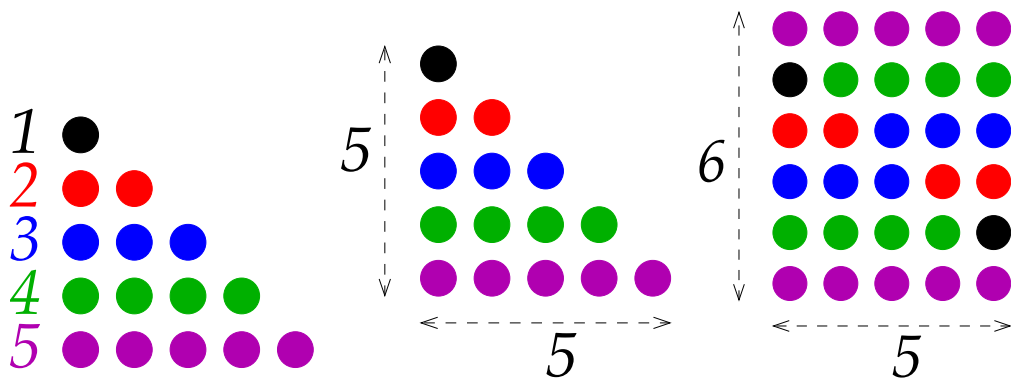
For example, let us sum the first five numbers.

$$1 + 2 + 3 + 4 + 5 = ?.$$

One possibility (not very efficient!) is to just sum one number after the other, and discover that

$$1 + 2 + 3 + 4 + 5 = 15.$$

More efficient and “geometric” idea, given in this animation:



1. Put the numbers in triangular form.
2. “Duplicate” the numbers, putting the last row on the top, and so on.
3. Find the size of the rectangle obtained in this way, and divide by two.

In this way

$$1 + 2 + 3 + 4 + 5 = 6 \times 5 : 2 = 15.$$

ACTIVITY: Calculate

$$1 + 2 + 3 + 4 + 5 + 6 = ?.$$

Do it in two ways:

1. *Knowing already that $1 + 2 + 3 + 4 + 5 = 15$ and summing the last number to this result,*
2. *Performing the computation from scratch, using the “geometric” idea explained above.*

ADDITIONAL ACTIVITY FOR GIFTED STUDENTS: Calculate

$$1 + 2 + 3 + 4 + 5 + 6 = ?$$

using the following strategy: sum the first number with the last one ($1 + 6$), the second number with the second-last one ($2 + 5$), the third number with the last but two one ($3 + 4$). Notice that you obtain always the same total (7) in summing these two numbers, hence, to obtain the final result, multiply it by the number of coupled numbers (3).

In this way:

$$1 + 2 + 3 + 4 + 5 + 6 = 7 * 3 = 21.$$

ADDITIONAL ACTIVITY FOR GIFTED STUDENTS: Calculate

$$1 + 2 + 3 + 4 + 5 + 6 + 7 = ?$$

ADDITIONAL ACTIVITY: Calculate

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10.$$

ADDITIONAL ACTIVITY FOR VERY GIFTED STUDENTS: Calculate the sum of the first 100 numbers (do not attempt to just sum the numbers one after the other, since it takes too much time!)...

$$1 + 2 + 3 + 4 + 5 + \cdots + 98 + 99 + 100 = ?.$$

(One possible solution is to match 1 with 100, then 2 with 99, then 3 with 98, ..., then 50 with 51, thus obtaining 50 pairs of numbers which sum to 101. In this way, one obtains the result $101 \times 50 = 5050$.)

Summing (and taking differences) in an efficient way may also boil down to detecting “hidden cancellations”.

For example, calculate

$$10 - 9 + 8 - 7 + 6 - 5 + 4 - 3 + 2 - 1 = ?.$$

Instead of doing the single operations one by one, it is more efficient to write the expression above as

$$(10 - 9) + (8 - 7) + (6 - 5) + (4 - 3) + (2 - 1).$$

In this way, we find five couples of numbers, grouped between the parentheses, whose difference is equal to 1. Then, we see that

$$10 - 9 + 8 - 7 + 6 - 5 + 4 - 3 + 2 - 1 = 5.$$

ACTIVITY: Calculate

$$100 - 99 + 98 - 97 + 96 - 95 + 94 - 93 + 92 - 91 + 90 - 89 = ?.$$

ACTIVITY: Calculate

$$10 - 7 + 9 - 6 + 8 - 5 + 7 - 4 + 6 - 3 + 5 - 2 + 4 - 1 = ?.$$

Hint: think about

$$(10 - 7) + (9 - 6) + (8 - 5) + (7 - 4) + (6 - 3) + (5 - 2) + (4 - 1).$$